

THE L^p -POINCARÉ INEQUALITY FOR ANALYTIC ORNSTEIN-UHLENBECK OPERATORS

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ABSTRACT. Consider the linear stochastic evolution equation

$$dU(t) = AU(t)dt + dW_H(t), \quad t \geq 0,$$

where A generates a C_0 -semigroup on a Banach space E and W_H is a cylindrical Brownian motion in a continuously embedded Hilbert subspace H of E . Under the assumption that the solutions to this equation admit an invariant measure μ_∞ we prove that if the associated Ornstein-Uhlenbeck semigroup is analytic and has compact resolvent, then the Poincaré inequality

$$\|f - \bar{f}\|_{L^p(E, \mu_\infty)} \leq \|D_H f\|_{L^p(E, \mu_\infty)}$$

holds for all $1 < p < \infty$. Here \bar{f} denotes the average of f with respect to μ_∞ and D_H the Fréchet derivative in the direction of H .

1. INTRODUCTION

Let E be a real Banach space and let H be a Hilbert subspace of E , with continuous embedding $i : H \hookrightarrow E$. Let A be the generator of a C_0 -semigroup $S = (S(t))_{t \geq 0}$ on E and let W_H be a cylindrical Brownian motion in H . Under the assumption that the linear stochastic evolution equation

$$(1.1) \quad dU(t) = AU(t) + dW_H(t), \quad t \geq 0,$$

has an invariant measure μ_∞ , we wish to establish sufficient conditions for the validity of the Poincaré inequality

$$\|f - \bar{f}\|_{L^p(E, \mu_\infty)} \leq C \|D_H f\|_{L^p(E, \mu_\infty)}, \quad 1 < p < \infty.$$

Here \bar{f} denotes the average of f with respect to μ_∞ and D_H the directional Fréchet derivative in the direction of H (see (2.4) below). To the best of our knowledge, this problem has been considered so far only for $p = 2$ and Hilbert spaces E . For this setting, Chojnowska-Michalik and Goldys [5] obtained various necessary and sufficient conditions for the inequality to be true. Here we show that these conditions are equivalent to another, formally weaker, condition and that these equivalent conditions imply the validity of the Poincaré inequality for all $1 < p < \infty$ (Theorem 2.4). Our proof depends crucially on the L^p -gradient estimates for analytic Ornstein-Uhlenbeck semigroups obtained in the recent papers [25, 26].

Related L^p -Poincaré inequalities have been proved in various other settings, e.g. for the classical Ornstein-Uhlenbeck semigroup (this corresponds to the case

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$A = -I$ of the setting considered here) [32, Eq. (2.5)], for the Walsh system [11], and in certain non-commutative situations [17, 35]. Poincaré inequalities are intimately related to other functional inequalities such as, log-Sobolev inequalities and transportation cost inequalities, and imply concentration-of-measure inequalities. For a comprehensive study of these topics we refer the reader to the recent monograph of Bakry, Gentil and Ledoux [1].

As an application of Theorem 2.4 we find that the L^p -Poincaré inequality holds if the Ornstein-Uhlenbeck semigroup P associated with (1.1) (see (2.1)) is analytic on $L^p(E, \mu_\infty)$ and has compact resolvent. In Section 3 we provide some examples in which the various assumptions are satisfied. In the final Section 4 we address the problem of compactness of certain tensor products of resolvents naturally associated with P .

All vector spaces are real. We will always identify Hilbert spaces with their dual via the Riesz representation theorem. The domain, kernel, and range of a linear operator A will be denoted by $D(A)$, $N(A)$, and $R(A)$, respectively. We write $a \lesssim b$ to mean that there exists a constant C , independent of a and b , such that $a \leq Cb$.

2. THE L^p -POINCARÉ INEQUALITY

Throughout this note we fix a Banach space E and a Hilbert subspace H of E , with continuous embedding $i : H \hookrightarrow E$, and make the following assumption.

Assumption 2.1. There exists a centred Gaussian Radon measure μ_∞ on E whose covariance operator $Q_\infty \in \mathcal{L}(E^*, E)$ is given by

$$\langle Q_\infty x^*, y^* \rangle = \int_0^\infty \langle QS^*(s)x^*, S^*(s)y^* \rangle ds, \quad x^*, y^* \in E^*.$$

Here $Q := i \circ i^*$; we identify H and its dual in the usual way. The convergence of the integrals on the right-hand side is part of the assumption. As is well known, Assumption 2.1 is equivalent to the existence of an invariant measure for the problem (1.1); we refer the reader to [10, 16] for the details. In fact, the measure μ_∞ is the minimal (in the sense of covariance domination) invariant measure for (1.1).

The formula

$$(2.1) \quad P(t)f(x) = \mathbb{E}(f(U(t, x))), \quad t \geq 0, x \in E,$$

where $U(t, x)$ denotes the unique mild solution of (1.1) with initial value x , defines a semigroup of linear contractions $P = (P(t))_{t \geq 0}$ on the space $B_b(E)$ of bounded real-valued Borel functions on E . This semigroup is called the *Ornstein-Uhlenbeck semigroup* associated with the pair (A, H) . By an easy application of Hölder's inequality, this semigroup extends uniquely to C_0 -semigroup of contractions on $L^p(E, \mu_\infty)$, which we shall also denote by P . Its generator will be denoted by L .

By a result of Chojnowska-Michalik and Goldys [4, 5] (see [28] for the formulation of this result in its present generality), the reproducing kernel Hilbert space H_∞ associated with the measure μ_∞ is invariant under the semigroup S and the restriction of S is a C_0 -semigroup of contractions on H_∞ . We shall denote this restricted semigroup by S_∞ and its generator by A_∞ . The inclusion mapping $H_\infty \hookrightarrow E$ will be denoted by i_∞ ; recall that $Q_\infty = i_\infty \circ i_\infty^*$ (see [16, 28]).

It has been shown in [4] (see also [28, 29]) that $P(t)$ is the so-called *second quantisation* of the adjoint semigroup $S_\infty^*(t)$. More precisely, the Wiener-Itô isometry establishes an isometric identification $L^2(E, \mu_\infty) = \bigoplus_{n \geq 0} H_\infty^{\otimes n}$, where $H_\infty^{\otimes n}$ is the

n -fold symmetric tensor product of H_∞ (the so-called n -th Wiener-Itô chaos), and under this isometry we have

$$P(t) = \bigoplus_{n \geq 0} S_\infty^{*\otimes n}(t).$$

We have $H_\infty^{\otimes 0} = \mathbb{R}\mathbf{1}$ (by definition) and $H_\infty^{\otimes 1} = H_\infty$. The latter identification allows us to deduce many properties of P from the corresponding properties of S_∞^* and vice versa and will be used freely in what follows.

Following [3, 16] we define \mathcal{F}^k as the space of all functions $f : E \rightarrow \mathbb{R}$ of the form

$$(2.2) \quad f(x) = \phi(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle)$$

for some $d \geq 1$, with $x_j^* \in E^*$ for all $j = 1, \dots, d$ with $\phi \in C_b^k(\mathbb{R}^d)$. Let

$$\mathcal{F}_A^k = \{f \in \mathcal{F}^k : x_j^* \in D(A^*) \text{ for all } j = 1, \dots, d \text{ and } \langle \cdot, A^* Df(\cdot) \rangle \in C_b(E)\}.$$

It follows from [3, 16] that \mathcal{F}_A^2 is a core for $D(L)$ in each $L^p(E, \mu_\infty)$ and that for $f, g \in \mathcal{F}_A^2$ we have the identity

$$(2.3) \quad \langle Lf, g \rangle + \langle Lg, f \rangle = - \int_E \langle D_H f, D_H g \rangle d\mu_\infty.$$

Here D_H denotes the Fréchet derivative in the direction of H , defined on \mathcal{F}^1 by

$$(2.4) \quad D_H f(x) := \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle) i^* x_j^*$$

with f and ϕ as in (2.2). It should be emphasised that D_H is not always closable; various conditions for closability as well as a counterexample are given in [15]. If P is analytic on $L^p(E, \mu_\infty)$ for some/all $1 < p < \infty$ (the equivalence being a consequence of the Stein interpolation theorem), then D_H is closable as an operator from $L^p(E, \mu_\infty)$ to $L^p(E, \mu_\infty; H)$ [16, Proposition 8.7].

The following necessary and sufficient condition for the L^2 -Poincaré inequality is essentially due to Chojnowska-Michalik and Goldys [6] (see also [9, Proposition 10.5.2]). Since the present formulation is slightly more general, for the convenience we include the proof which follows the lines of [6].

Proposition 2.2 (Poincaré inequality, the case $p = 2$). *Let Assumption 2.1 hold and fix a number $\omega > 0$. If D_H is closable as a densely defined operator in $L^2(E, \mu_\infty)$, then the following assertions are equivalent:*

- (1) $\|S_\infty(t)\| \leq e^{-\omega t}$ for all $t \geq 0$;
- (2) *The Poincaré inequality*

$$\|f - \bar{f}\|_{L^2(E, \mu_\infty)} \leq \frac{1}{\sqrt{2\omega}} \|D_H f\|_{L^2(E, \mu_\infty)}, \quad f \in D(D_H),$$

holds. Here, $\bar{f} = \int_E f d\mu_\infty$.

Proof. (1) \Rightarrow (2): Since $t \mapsto e^{\omega t} S_\infty^*(t)$ is a C_0 -contraction semigroup, by second quantisation the same is true for the direct sum for $n \geq 1$ of their n -fold symmetric tensor products, $\bigoplus_{n \geq 1} e^{n\omega t} S_\infty^{*\otimes n}(t)$. Replacing $e^{n\omega t}$ by $e^{\omega t}$, the resulting direct sum $\bigoplus_{n \geq 1} e^{\omega t} S_\infty^{*\otimes n}(t)$ is contractive as well. This semigroup is generated by the part

$L_0 + \omega$ of $L + \omega$ in $L_0^2(E, \mu_\infty) := L^2(E, \mu_\infty) \ominus \mathbb{R}1$. Thus we obtain the dissipativity inequality

$$-\langle (L_0 + \omega)f, f \rangle \geq 0, \quad f \in \mathcal{D}(L_0).$$

In view of (2.3), this gives the inequality

$$\omega \|f\|_2^2 \leq -\langle L_0 f, f \rangle = \frac{1}{2} \|D_H f\|_2^2, \quad f \in \mathcal{D}(L_0) \cap \mathcal{F}_A^2.$$

As a consequence,

$$(2.5) \quad \omega \|f - \bar{f}\|_2^2 \leq \frac{1}{2} \|D_H f\|_2^2, \quad f \in \mathcal{F}_A^2.$$

It is routine (albeit somewhat tedious) to check that the inequality (2.5) extends to $f \in \mathcal{F}^1$, and since by definition this is a core for $\mathcal{D}(D_H)$ it extends to arbitrary elements $g \in \mathcal{D}(D_H)$.

(2) \Rightarrow (1): Every $x^* \in E^*$, when viewed as an element of $L^2(E, \mu_\infty)$, satisfies $D_H x^* = i^* x^*$. Moreover, if $x^* \in \mathcal{D}(A^*)$, then $A_\infty^* x^* \in \mathcal{D}(A_\infty^*)$, and therefore (identifying H_∞ with the first Wiener-Itô chaos) $x^* \in \mathcal{D}(L)$ as an element of $L^2(E, \mu_\infty)$.

By specialising the Poincaré inequality to functionals x^* we obtain the inequality

$$\|i_\infty^* x^*\| = \|x^*\|_{L^2(E, \mu_\infty)} \leq \frac{1}{\sqrt{2\omega}} \|i^* x^*\|, \quad x^* \in \mathcal{D}(A^*).$$

In the same way, (2.3) takes the form

$$\langle A_\infty^* i_\infty^* x^*, i_\infty^* x^* \rangle = -\frac{1}{2} \|i^* x^*\|^2, \quad x^* \in \mathcal{D}(A^*).$$

Combining these inequalities, we obtain

$$-\langle A_\infty^* i_\infty^* x^*, i_\infty^* x^* \rangle \geq \omega \|i_\infty^* x^*\|^2, \quad x^* \in \mathcal{D}(A^*).$$

Since the elements $i_\infty^* x^*$ with $x^* \in \mathcal{D}(A^*)$ form a core for $\mathcal{D}(A_\infty^*)$, this is equivalent to saying that $A_\infty^* + \omega$ is dissipative on H_∞ . It follows that $\|S_\infty^*(t)\| \leq \exp(-\omega t)$ for all $t \geq 0$. \square

The main result of this note (Theorem 2.4) asserts that if P is analytic and A_∞^* has closed range, then all conditions of Proposition 2.2 are satisfied and the Poincaré inequality extends to $L^p(E, \mu_\infty)$ for all $1 < p < \infty$. To prepare for the proof we need to recall some preliminary facts. We begin by imposing the following assumption, which will be in force for the rest of this section.

Assumption 2.3. For some (equivalently, for all) $1 < p < \infty$ the semigroup P extends to an analytic C_0 -semigroup on $L^p(E, \mu_\infty)$.

The problem of analyticity of P has been studied in several articles [13, 14, 16, 24, 26]. In these, necessary and sufficient conditions for analyticity can be found. We have already mentioned the fact that if P is analytic on $L^p(E, \mu_\infty)$ for some/all $1 < p < \infty$, then D_H is closable as an operator from $L^p(E, \mu_\infty)$ to $L^p(E, \mu_\infty; H)$. In what follows, D_H will always denote this closure and $\mathcal{D}(D_H)$ its domain in $L^p(E, \mu)$. Note that there is a slight abuse of notation here, as $\mathcal{D}(D_H)$ obviously depends on p . The choice of p will always be clear from the context, and for this reason we prefer not to overburden notations. The same slight abuse of notation applies to the notation $\mathcal{D}(L)$ for the domain of L in $L^p(E, \mu_\infty)$.

From [24] we know that if P is analytic, then the generator L of P can be represented as

$$(2.6) \quad L = D_H^* B D_H$$

for a unique bounded operator B on H which satisfies

$$B + B^* = -I.$$

The rigorous interpretation of (2.6) is that for $p = 2$ the operator $-L$ is the sectorial operator associated with the closed continuous accretive form

$$(f, g) \mapsto -\langle B D_H f, D_H g \rangle.$$

In the sequel we will use the standard fact (which is proved by hypercontractivity arguments) that for each $n \geq 0$ the summand $H_\infty^{\otimes n}$ in the Wiener-Itô decomposition for $L^2(E, \mu_\infty)$ is contained as a closed subspace in $L^p(E, \mu_\infty)$ for all $1 < p < \infty$. In view of this we will continue to refer to $H_\infty^{\otimes n}$ as the n -th Wiener chaos. By an interpolating argument (see [29, Lemma 4.2]) we obtain the estimate $\|P(t)\|_p \leq \|S_\infty(t)\|^{n\theta_p}$ on each of these subspaces, with a constant $0 < \theta_p < 1$ depending only on p . Summing over $n \geq 1$ and passing to the closure of the linear span, we obtain the estimate

$$(2.7) \quad \|P(t)\|_p \leq \|S_\infty(t)\|^{\theta_p} \text{ on } L^p(E, \mu_\infty) \ominus \mathbb{R}\mathbf{1}.$$

Theorem 2.4 (L^p -Poincaré inequality). *Let Assumptions 2.1 and 2.3 hold. Then the following assertions are equivalent:*

- (1) A_∞^* has closed range;
- (2) there exists $\omega > 0$ such that $\|S_\infty(t)\| \leq e^{-\omega t}$ for all $t \geq 0$;
- (3) there exist $M \geq 1$ and $\omega > 0$ such that $\|S_\infty(t)\| \leq M e^{-\omega t}$ for all $t \geq 0$;
- (4) there exist $M \geq 1$ and $\omega > 0$ such that $\|S_H(t)\| \leq M e^{-\omega t}$ for all $t \geq 0$;
- (5) H_∞ embeds continuously in H ;
- (6) for some $1 < p < \infty$ there exists a finite constant $C \geq 0$ such that

$$\|f - \bar{f}\|_{L^p(E, \mu_\infty)} \leq C_p \|D_H f\|_{L^p(E, \mu_\infty)}, \quad f \in D(D_H);$$

- (7) for all $1 < p < \infty$ there exists a finite constant $C \geq 0$ such that

$$\|f - \bar{f}\|_{L^p(E, \mu_\infty)} \leq C_p \|D_H f\|_{L^p(E, \mu_\infty)}, \quad f \in D(D_H).$$

In what follows we will say that the L^p -Poincaré inequality holds if condition (7) is satisfied.

Before we start with the proof we recall some further useful facts. Firstly, on the first Wiener chaos, (2.6) reduces to the identity

$$A_\infty^* = V^* B V,$$

where V is the closure of the mapping $i_\infty^* x^* \mapsto i^* x^*$; see [15, 25, 26]. Secondly, in [26] it is shown that Assumption 2.3 implies that S maps H into itself and that its restriction to H extends to a bounded analytic C_0 -semigroup on H . We shall denote this semigroup by S_H and its generator by A_H .

Proof of Theorem 2.4. (1) \Rightarrow (3): Let us first observe that the strong stability of S_∞^* [16, Proposition 2.4] implies that $N(A_\infty^*) = \{0\}$.

Suppose next that some $h \in H_\infty$ annihilates the range of A_∞^* . As $\langle A_\infty^* g, h \rangle = \langle V^* B V g, h \rangle = 0$ for all $g \in D(A_\infty^*)$, it follows that $h \in D(V)$ and $\langle B V^* g, V h \rangle = 0$ for all $g \in D(A_\infty^*)$. Using that $D(A_\infty^*)$ is a core for $D(V)$ (see [25]), it follows that

$\langle BV^*g, Vh \rangle = 0$ for all $g \in \mathcal{D}(V)$. In particular, $\langle BV^*h, Vh \rangle = 0$. Since also $\langle BV^*h, Vh \rangle = -\frac{1}{2}\|Vh\|^2$ by the identity $B + B^* = -I$, it follows that $Vh = 0$ and therefore $h \in \mathcal{N}(A_\infty^*) = \mathcal{N}(V)$. But we have already seen that $\mathcal{N}(A_\infty^*) = \{0\}$ and we conclude that $h = 0$.

This argument proves that $\mathcal{R}(A_\infty^*)$ is dense. Since by assumption A_∞^* has closed range, it follows that A_∞^* is surjective. As we observed at the beginning of the proof, A_∞^* is also injective, and therefore A_∞^* is boundedly invertible by the closed graph theorem. Since A_∞^* generates an analytic C_0 -contraction semigroup, the spectral mapping theorem for analytic C_0 -semigroups (see [12]) implies that S_∞^* is uniformly exponentially stable.

(3) \Rightarrow (7): Fix an arbitrary $1 < p < \infty$. Fix a function $f \in \mathcal{F}^0$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|f - \bar{f}\| = \sup_{\substack{\|g\|_q \leq 1 \\ \bar{g} = 0}} |\langle f - \bar{f}, g \rangle| = \sup_{\substack{\|g\|_q \leq 1 \\ \bar{g} = 0}} |\langle f - \bar{f}, g - \bar{g} \rangle| = \sup_{\substack{\|g\|_q \leq 1 \\ \bar{g} = 0}} |\langle f, g - \bar{g} \rangle|,$$

where it suffices to consider functions $g \in \mathcal{F}^0$. Next we observe that, by (2.7),

$$\langle f, g - \bar{g} \rangle = \lim_{t \rightarrow \infty} \langle f, g - P(t)g \rangle.$$

Following an argument in [22, Lemma 3] we have

$$\langle f, g - P(t)g \rangle = - \int_0^t \langle f, LP(s)g \rangle ds = - \int_0^t \langle D_H f, BD_H P(s)g \rangle ds.$$

If in addition $\bar{g} = 0$ (i.e. if $g \in L^p(E, \mu_\infty) \ominus \mathbb{R}\mathbf{1}$), then for all $t \geq 1$ we have

$$\begin{aligned} |\langle f, g - P(t)g \rangle| &\leq \|B\| \|D_H f\|_p \left(\int_0^1 + \int_1^\infty \right) \|D_H P(s)g\|_q ds \\ &\lesssim \|D_H f\|_p \left(\int_0^1 \frac{1}{\sqrt{s}} \|g\|_q ds + \|D_H P(1)\| \int_0^\infty e^{-\omega\theta_q} \|g\|_q ds \right). \end{aligned}$$

where we used the gradient estimates of [25] and (2.7). Taking the supremum over all $g \in \mathcal{F}^0$ of L^q -norm 1 with $\bar{g} = 0$, this gives

$$\|f - \bar{f}\|_p \lesssim \|D_H f\|_p.$$

Since \mathcal{F}^0 is a core for $\mathcal{D}(D_H)$ this concludes the proof of the implication.

(7) \Rightarrow (6): This implication is trivial.

(6) \Rightarrow (3): This follows from Proposition 2.2 along with the fact that H_∞ is isomorphic to the first Wiener-Itô chaos in $L^p(E, \mu_\infty)$.

(3) \Rightarrow (1): The uniform exponential stability of S_∞^* implies that A_∞^* is boundedly invertible.

(3) \Leftrightarrow (4) \Leftrightarrow (5): These equivalences have been proved in [16, Theorem 5.4].

(7) \Rightarrow (2): This follows from Proposition 2.2.

(2) \Rightarrow (3): Trivial. \square

The equivalent conditions of the theorem do not in general imply the existence of an $\omega > 0$ such that $\|S_H(t)\| \leq e^{-\omega t}$ for all $t \geq 0$:

Example 2.5. Consider the Dirichlet Laplacian Δ on $E = L^2(-1, 1)$ and take $H = E$. Let S denote the heat semigroup generated by Δ on this space. Fix $\omega > 0$. As is well known and easy to check, Assumptions 2.1 and 2.3 are satisfied for the operator

$\Delta - \omega$. Let us now replace the norm of $L^2(-1, 1)$ by the equivalent (Hilbertian) norm

$$\|f\|_{(r)}^2 := \|f|_{(-1,0)}\|^2 + r^2 \|f|_{(0,1)}\|^2,$$

where $r > 0$ is a positive scalar. Starting from an initial condition with support in $(-1, 0)$, the semigroup $s_\omega(t) = e^{-\omega t} S(t)$ generated by $\Delta - \omega$ will instantaneously spread out the support of f over the entire interval $(-1, 1)$. Hence if we fix $t_0 > 0$ and $\omega > 0$ we may choose $r_0 > 0$ so large that

$$\|S_\omega(t_0)f\|_{(r)} > \|f\|_{(r)}.$$

As a result, the semigroup S_ω is uniformly exponentially stable but not contractive on $L^2(-1, 1)$ endowed with the norm $\|\cdot\|_{(r_0)}$.

One could object to this example that there is an equivalent Hilbertian norm (namely, the original norm of $L^2(-1, 1)$) on which we do have $\|S_\omega(t)\| \leq e^{-\omega t}$. There exist examples, however, of bounded analytic Hilbert space semigroups which are not similar to an analytic contraction semigroup. Such examples may be realised as multiplication semigroups on a suitable (pathological) Schauder basis (see, e.g., [21] and the references given there). For such examples, Assumptions 2.1 and 2.3 are again satisfied and we obtain a counterexample that cannot be repaired by a Hilbertian renorming.

As an application of Theorem 2.4 we have the following sufficient condition for the validity of the L^p -Poincaré inequality.

Theorem 2.6 (Compactness implies the L^p -Poincaré inequality). *Let Assumptions 2.1 and 2.3 hold and fix $1 < p < \infty$. The following assertions are equivalent:*

- (1) L has compact resolvent on $L^p(E, \mu_\infty)$;
- (2) P is compact on $L^p(E, \mu_\infty)$;
- (3) A_∞ has compact resolvent on H_∞ ;
- (4) S_∞ is compact on H_∞ ;
- (5) A_H has compact resolvent on H ;
- (6) S_H is compact on H .

If these equivalent conditions are satisfied, then the L^p -Poincaré inequality holds for all $1 < p < \infty$.

Proof. The equivalences (1) \Leftrightarrow (2), (3) \Leftrightarrow (4), and (5) \Leftrightarrow (6) follow from [12, Theorem 4.29] since P , S_∞ , and S_H are analytic semigroups.

We will prove next that (4) implies the validity of the L^p -Poincaré inequality. We will use some elementary facts from semigroup theory which can all be found in [12]. The strong stability of S_∞^* implies that 1 is not an eigenvalue of $S_\infty^*(t)$ for any $t > 0$. Since these operators are compact it follows that $1 \notin \sigma(S_\infty^*(t))$, which in turn implies that $0 \notin \sigma(A_\infty^*)$ by the spectral mapping theorem for eventually norm continuous semigroups. By the equality spectral bound and growth bound for such semigroups, it follows that S_∞^* (and hence also S_∞) is uniformly exponentially stable. We may now apply Theorem 2.4 to obtain the conclusion.

(2) \Rightarrow (4): This follows by restricting to the first Wiener-Itô chaos.

(4) \Rightarrow (2): We have already seen that (4) implies that S_∞^* is uniformly exponentially stable. Because of this, the compactness of $S_\infty^*(t)$ implies, by second quantisation, the compactness of $P(t)$ on $L^p(E, \mu_\infty)$ (cf. [29, Lemma 4.2]).

(4) \Rightarrow (6): By [16, Theorem 3.5] combined with [28, Proposition 1.3], for each $t > 0$ the operator $S(t)$ maps H into H_∞ ; we shall denote this operator by $S_{H,\infty}(t)$.

Furthermore we have a continuous embedding $i_{\infty, H} : H_{\infty} \hookrightarrow H$ [16, Theorem 5.4] (this result can be applied here since, by what has already been proved, (2) implies the uniform exponential stability of S_{∞}). Now if S_{∞} is compact, the compactness of S_H follows from the factorisation

$$S_H(t) = i_{\infty, H} \circ S_{\infty}(t/2) \circ S_{H, \infty}(t/2).$$

(6) \Rightarrow (4): We will show that (6) implies that H_{∞} embeds into H . Once we know this, (4) follows from the factorisation $S_{\infty}(t) = S_{H, \infty}(t/2) \circ S_H(t/2) \circ i_{\infty, H}$.

This concludes the proof of the equivalences of the conditions (1)–(6). To complete the proof we will now show that these conditions imply the validity of the Poincaré inequality.

Suppose that $h \in H$ is a vector satisfying $S_H(t)h = h$ for all $t \geq 0$. Since $S(t)$ maps H into H_{∞} (see [16, Proposition 2.3]) this means that $h \in H_{\infty}$. But then in E for all $t \geq 0$ we have $i_{\infty} S_{\infty}(t)h = i_H S_H(t)h = i_H h = i_{\infty} h$, so that in H_{∞} we obtain $S_{\infty}(t)h = h$ for all $t \geq 0$. Hence, for all $h' \in H_{\infty}$,

$$\langle h, h' \rangle_{H_{\infty}} = \lim_{t \rightarrow \infty} \langle S_{\infty}(t)h, h' \rangle_{H_{\infty}} = \lim_{t \rightarrow \infty} \langle h, S_{\infty}^*(t)h' \rangle_{H_{\infty}} = 0$$

by the strong stability of S_{∞}^* . This being true for all $h' \in H_{\infty}$, it follows that $h = 0$. We have thus shown that 1 is not an eigenvalue of $S_H(t)$. Having arrived at this conclusion, the argument given above for S_{∞} can now be repeated to conclude that S_H is uniformly exponentially stable. Now Theorem 2.4 implies that H_{∞} embeds into H . \square

Remark 2.7. The equivalence of (4) and (6) for symmetric Ornstein-Uhlenbeck semigroups follows from [8, Theorem 2.9].

Corollary 2.8. *Let $1 < p < \infty$. If the embedding $D(D_H) \hookrightarrow L^p(E, \mu_{\infty})$ is compact, then the L^p -Poincaré inequality holds.*

Recall our abuse of notation to denote by $D(D_H)$ and $D(L)$ the domains of closed operators D_H and L in $L^p(E, \mu_{\infty})$. Necessary and sufficient conditions for the compactness of the embedding $D(D_H) \hookrightarrow L^p(E, \mu_{\infty})$ are stated in [15].

Proof. Since $D(L)$ embeds into $D(D_H)$ (see [25, Theorem 8.2]) this is immediate from the previous theorem. \square

Our next aim is to show that also an L^p -inequality holds for the adjoint operator D_H^* . Here we view D_H as a closed densely defined operator from $L^q(E, \mu_{\infty})$ into $\overline{R(D_H)}$ and D_H^* a closed densely defined operator from $\overline{R(D_H)}$ into $L^p(E, \mu_{\infty})$, $\frac{1}{p} + \frac{1}{q} = 1$. The proof relies on some facts that have been proved in [25, 26]. We start by observing that if Assumptions 2.1 and 2.3 hold, then the semigroup

$$\underline{P}(t) := P(t) \otimes S_H^*(t)$$

extends to a bounded analytic C_0 -semigroup on $L^p(E, \mu_{\infty}; H)$, $1 < p < \infty$. We will need the fact that on $\overline{R(D_H)}$ the generator \underline{L} of this semigroup is given by

$$\underline{L} = D_H D_H^* B;$$

the proof as well as the rigorous interpretation of the right-hand side is given in the references just quoted.

Theorem 2.9 (L^p -Poincaré inequality for D_H^*). *Let Assumptions 2.1 and 2.3 hold. If the equivalent conditions of Theorem 2.4 are satisfied, then there exists a finite constant $C \geq 0$ such that for all $1 < p < \infty$ we have*

$$\|f\|_{L^p(E, \mu_\infty; H)} \leq C_p \|D_H^* f\|_{L^p(E, \mu_\infty; H)}, \quad f \in \mathcal{D}(D_H^*),$$

where D_H^* is interpreted as explained above.

Proof. We can follow the proof of Theorem 2.4, this time using that for bounded cylindrical functions $f, g \in \overline{\mathcal{R}(D_H)}$ we have

$$\langle f, g - \underline{P}(t)g \rangle = - \int_0^t \langle f, \underline{L}\underline{P}(s)g \rangle ds = - \int_0^t \langle D_H^* f, D_H^* B \underline{P}(s)g \rangle ds.$$

For $t \geq 1$ we then have

$$\begin{aligned} |\langle f, g - \underline{P}(t)g \rangle| &\leq \|B\| \|D_H^* f\|_p \left(\int_0^1 + \int_1^\infty \right) \|D_H^* B \underline{P}(s)g\|_q ds \\ &\lesssim \|D_H^* f\|_p \left(\int_0^1 \frac{1}{\sqrt{s}} \|g\|_q ds + \|D_H^* B \underline{P}(1)\| \int_0^\infty e^{-\omega \theta_q} \|g\|_q ds \right), \end{aligned}$$

this time using the gradient estimates for $D_H^* B$ (cf. the proof of [25, Proposition 9.3] where resolvents are used instead of the semigroup operators) and the uniform exponential stability of $\underline{P} = P \otimes S_H^*$. The proof can be finished along the lines of Theorem 2.4; this time we use that $\lim_{t \rightarrow \infty} \langle f, g - \underline{P}(t)g \rangle = \langle f, g \rangle$. \square

3. EXAMPLES

Example 3.1 (Finite dimensions and non-degenerate noise). Suppose that $H = E = \mathbb{R}^d$ and let Assumption 2.1 hold. Then $H^\infty = \mathbb{R}^d$. Under these assumptions, a result of Fuhrman [13, Theorem 3.6 and Corollary 3.8] implies that Assumption 2.3 holds. By finite-dimensionality, the conditions of Theorems 2.4 and 2.9 are satisfied. It follows that the L^p -Poincaré inequalities for D_H and D_H^* hold for $1 < p < \infty$.

Example 3.2 (The self-adjoint case). Suppose that $H = E$ and S is self-adjoint on E . Then Assumption 2.1 holds if and only if S is uniformly exponentially stable. In this situation, by [16] also S_∞ is self-adjoint and uniformly exponentially stable, and P is self-adjoint on $L^2(E, \mu_\infty)$. In particular, Assumption 2.3 then holds and therefore the equivalent conditions of Theorem 2.4 are satisfied. It follows that the L^p -Poincaré inequality holds for $1 < p < \infty$.

Example 3.3 (The strong Feller case). Suppose that Assumptions 2.1 and 2.3 hold, and that P is strongly Feller. As is well known, this is equivalent to the condition that for each $t > 0$ the semigroup operator $S(t)$ maps E into the reproducing kernel Hilbert space H_t associated with μ_t , the centred Gaussian Radon measure on E whose covariance operator $Q_t \in \mathcal{L}(E^*, E)$ is given by

$$\langle Q_t x^*, y^* \rangle = \int_0^t \langle Q S^*(s) x^*, S^*(s) y^* \rangle ds, \quad x^*, y^* \in E^*.$$

These measures exist by a standard covariance domination argument (note that $\langle Q_t x^*, x^* \rangle \leq \langle Q_\infty x^*, x^* \rangle$). By [28] we have a contractive embedding $i_{t,\infty} : H_t \hookrightarrow H_\infty$. Then $S_\infty(t) = i_{t,\infty} \circ S(t) \circ i_\infty$, where $i_\infty : H_\infty \hookrightarrow E$ is the inclusion mapping. The compactness of $i_\infty : H_\infty \hookrightarrow E$ (this mapping being γ -radonifying; see [30]) implies that $S_\infty(t)$ is compact for all $t > 0$, and by a general result from semigroup theory this implies that the resolvent operators $R(\lambda, A_\infty)$ are compact. Similarly

from $S_H(t) = i_{t,\infty} i_{\infty,H} \circ S(t) \circ i_\infty$, where $i_{\infty,H} : H_\infty \hookrightarrow H$ is the embedding mapping (see [16, Theorem 5.4] for the proof that this inclusion holds under the present assumptions) it follows that $S_H(t)$ is compact and therefore $R(\lambda, A_H)$ are compact. It follows that the L^p -Poincaré inequalities for D_H and D_H^* hold for $1 < p < \infty$.

Example 3.4 (The case $D(A) \hookrightarrow H_\infty$). Suppose that Assumptions 2.1 and 2.3 hold, and that we have a continuous inclusion $D(A) \hookrightarrow H_\infty$. Then $R(\lambda, A_\infty) = i_A R(\lambda, A) i_\infty$, where $i_\infty : H_\infty \hookrightarrow E$ and $i_A : D(A) \hookrightarrow H_\infty$ are the inclusion mappings. The compactness of $i_\infty : H_\infty \hookrightarrow E$ implies that $R(\lambda, A_\infty)$ is compact. It follows that the L^p -Poincaré inequality for D_H holds for $1 < p < \infty$. A similar argument (using again that $H_\infty \hookrightarrow H$) shows that if the inclusion $H \hookrightarrow E$ is compact, then $R(\lambda, A_H)$ is compact as well and the L^p -Poincaré inequalities for D_H and D_H^* hold for $1 < p < \infty$.

In fact the same results hold if $D(A^n) \hookrightarrow H_\infty$ for some large enough $n \geq 1$. We give the argument for $n = 2$; it is clear from this argument that we may proceed inductively to prove the general case. For $n = 2$ we repeat the above proof we now obtain $\mu R(\mu, A_\infty) R(\lambda, A_\infty) = \mu i_{A^2} R(\mu, A) R(\lambda, A) i_\infty$, where $i_\infty : H_\infty \hookrightarrow E$ and $i_{A^2} : D(A^2) \hookrightarrow H_\infty$ are the inclusion mappings. It follows that $\mu R(\mu, A_\infty) R(\lambda, A_\infty)$ is compact for each $\mu \in \varrho(A_\infty)$. Passing to the limit $\mu \rightarrow \infty$, noting that by the resolvent identity we have

$$\begin{aligned} & \left\| \mu R(\mu, A_\infty) R(\lambda, A_\infty) - R(\lambda, A_\infty) \right\| \\ &= \left\| \frac{\mu}{\mu - \lambda} (R(\lambda, A_\infty) - R(\mu, A_\infty)) - R(\lambda, A_\infty) \right\| \\ &\leq \left\| \frac{\mu}{\mu - \lambda} R(\mu, A_\infty) \right\| + \left\| \left(\frac{\mu}{\mu - \lambda} - 1 \right) R(\lambda, A_\infty) \right\|, \end{aligned}$$

and using that $\|R(\nu, A_\infty)\| \leq 1/\nu$, it follows that $R(\lambda, A_\infty)$ is compact, being the uniform limit of compact operators.

4. COMPACTNESS RESULTS

In [5], a condition equivalent to the Poincaré inequality has been used to prove, under an additional Hilbert-Schmidt assumption, the compactness of the semigroup $P \otimes S_H^*$ on $L^p(E, \mu_\infty; H)$. The importance of this semigroup is apparent from the proof of Theorem 2.9 and the results in [5, 7, 25, 26] where this semigroup plays a crucial rôle in identifying the domains of $\sqrt{-L}$ and L . Here we wish to show that the compactness of this semigroup and its resolvent can be deduced under quite minimal assumptions.

We begin with a lemma which is based on the classical result of Paley [31] and Marcinkiewicz and Zygmund [27] (see also [33]) that if T is a bounded operator on a space $L^p(\nu)$ and if H is a Hilbert space, then $T \otimes I$ is bounded on $L^p(\nu; H)$ and $\|T \otimes I\| = \|T\|$. As a direct consequence, if S is a bounded operator on H , then $T \otimes S = (T \otimes I) \circ (I \otimes S)$ is bounded on $L^p(\nu; H)$ and $\|T \otimes S\| \leq \|T\| \|S\|$.

Lemma 4.1. *Let $1 \leq p < \infty$. If T is compact on $L^p(\nu)$ and S is compact on H , then $T \otimes S$ is compact on $L^p(\nu; H)$.*

Proof. Since compactness can be tested sequentially, there is no loss of generality in assuming that both $L^p(\nu)$ and H are separable. Since separable spaces $L^p(\nu)$ have the approximation property, by [23, Theorem 1.e.4] there is a finite rank operator

T' on $L^p(\nu)$ such that $\|T - T'\| < \varepsilon$. Similarly there is a finite rank operator S' on H such that $\|S - S'\| < \varepsilon$. Then $T' \otimes S'$ is a finite rank operator on $L^p(\nu; H)$ and

$$\|T' \otimes S' - T \otimes S\| \leq \|T' \otimes (S' - S)\| + \|(T' - T) \otimes S\| \leq \varepsilon(\|T\| + \varepsilon) + \|S\|.$$

This proves that $T \otimes S$ can be uniformly approximated by finite rank operators. \square

We now return to the setting of the previous section. Since a semigroup which is norm continuous for $t > 0$ is compact for $t > 0$ if and only if its resolvent operators are compact, Lemma 4.1 implies:

Proposition 4.2. *Let $1 < p < \infty$ and suppose that Assumptions 2.1 and 2.3 hold. If P has compact resolvent on $L^p(E, \mu_\infty)$, then $P \otimes S_H^*$ has compact resolvent on $L^p(E, \mu_\infty; H)$.*

The generator of $P \otimes S_H^*$ equals $L \otimes I + I \otimes A_H^*$. As we have seen, the compactness of the resolvent of L implies the compactness of the resolvent A_H^* . Thus the proposition suggests the more general problem whether $A \otimes I + I \otimes B$ has compact resolvent if A and B have compact resolvents. Our final result gives an affirmative answer for sectorial operators A and B of angle $< \frac{1}{2}\pi$. Recall that a densely defined closed linear operator A is said to be *sectorial operator of angle* $< \frac{1}{2}\pi$ if there exists an angle $0 < \theta < \frac{1}{2}\pi$ such that $\{|\arg z| > \theta\} \subseteq \varrho(A)$ and $\sup_{\{|\arg z| > \theta\}} \|zR(z, A)\| < \infty$.

Proposition 4.3. *Let $1 \leq p < \infty$ and suppose that A and B are sectorial operators of angle $< \frac{1}{2}\pi$ on $L^p(\nu)$ and H , respectively. If, for some $w_0 \in \varrho(A)$ and $z_0 \in \varrho(A)$, the operators $R(w_0, A)$ and $R(z_0, B)$ are compact, then $A \otimes I + I \otimes B$ has compact resolvent on $L^p(\nu; H)$.*

Proof. Fix numbers $\omega_A < \theta_A < \frac{1}{2}\pi$, $\omega_B < \theta_B < \frac{1}{2}\pi$, where ω_A and ω_B denote the angles of sectoriality of A and B . Fix $\lambda \in \mathbb{C}$ with $|\arg \lambda| > \theta$ and fix a number $0 < r < |\lambda|$. Let $\gamma_{A,r}$ and $\gamma_{B,r}$ be the downwards oriented boundaries of $\{|z| < r\} \cup \{|\arg z| < \theta_A\}$ and $\{|z| < r\} \cup \{|\arg z| < \theta_B\}$. It follows from [18, Formulas (2.2), (2.3)] and a limiting argument that

$$(4.1) \quad R(\lambda, A \otimes I + B \otimes I) = \frac{1}{(2\pi i)^2} \int_{\gamma_{B,r}} \int_{\gamma_{A,r}} \frac{1}{\lambda - (w + z)} R(w, A) \otimes R(z, B) dw dz;$$

note that the double integral on the right-hand side converges absolutely.

Given $\varepsilon > 0$ fix $R > r$ so large that

$$\left\| \frac{1}{(2\pi i)^2} \int_{\gamma_{B,r} \cap \mathbb{C}_{B_R}} \int_{\gamma_{A,r} \cap \mathbb{C}_{B_R}} \frac{1}{\lambda - (w + z)} R(w, A) \otimes R(z, B) dw dz \right\| < \varepsilon,$$

where $B_R = \{z \in \mathbb{C} : |z| < R\}$ and \mathbb{C}_{B_R} is its complement. By Lemma 4.1 and [34, Theorem 1.3] the operator

$$\frac{1}{(2\pi i)^2} \int_{\gamma_{B,r} \cap B_R} \int_{\gamma_{A,r} \cap B_R} \frac{1}{\lambda - (w + z)} R(w, A) \otimes R(z, B) dw dz$$

is compact, as it is the strong integral over a finite measure space of an integrand with values in the space of compact operators. As a consequence, for each $\varepsilon > 0$ we obtain that $R(\lambda, A \otimes I + B \otimes I) = K_\varepsilon + L_\varepsilon$ with K_ε compact and L_ε bounded with $\|L_\varepsilon\| < \varepsilon$. It follows that the range of the unit ball of $L^p(\nu; H)$ under $R(\lambda, A \otimes I + I \otimes B)$ is totally bounded and therefore relatively compact. \square

The formula (4.1) for the resolvent of the sum of two operators goes back to Bianchi and Favella [2] who considered bounded A and B . It can be viewed as a special instance of the so-called joint functional calculus for sectorial operators; see [20, Theorem 2.2], [19, Theorem 12.12].

Remark 4.4. The above proof easily extends to tensor products of C_0 -semigroups on arbitrary Banach spaces, provided one makes appropriate assumptions on the boundedness of the tensor products of the various bounded operators involved.

Remark 4.5. The same proof may be used to see that if A and B are resolvent commuting sectorial operators of angle $< \frac{1}{2}\pi$ on a Banach space X and if, for some $w_0 \in \varrho(A)$ and $z_0 \in \varrho(B)$, the operator $R(w_0, A)R(z_0, B)$ is compact on X , then $A + B$ has compact resolvent on X .

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